Indices and Surds

Type – I

⟹ Let n be a positive integer and a be real number, then:

$a^n = \frac{a \times a \times a \times \ldots\ldots \times a}{(n\ factors)}$

a^n is called "n[th] Power of a" or

"a raised to the power n"

where, a is called the base and n is called index or exponent of the power a^n.

E.x. 3^2 = square of 3, 3^3 = cube of 3 etc.

Laws of Indices:

1. $a^m \times a^n = a^{m+n}$ $where\ a \neq 0\ and\ (m,n) \in I$
2. $a^n \times a^n \times a^p \times \ldots\ldots\ldots.. = a^{m+n+p\ldots\ldots\ldots}$
3. $\frac{a^m}{a^n} = \{\frac{a^{m-n}\ 1}{1^{n-m}}$ $\begin{matrix} if\ m > n \\ if\ n > m \\ if\ m = n \end{matrix}$
4. $(a^m)^n = a^{nm} = (a^n)^m$
5. $a^{m^n} = a^{m \times m \times \ldots\ldots n}$ $times \neq (a^m)^n$
6. $(ab)^n = a^n b^n$
7. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$
8. $(-a)^n = \begin{bmatrix} a^n, when\ n\ is\ even \\ -a^n, when\ n\ is\ odd \end{bmatrix}$

These rules are also true when n is negative or fraction.

9. $a^n = a^{(-1)n} = (a^{-1})^n = \left(\frac{1}{a}\right)^n$

$= \frac{1}{a} \times \frac{1}{a} \times \frac{1}{a} \ldots\ldots\ldots n$ times

10. $a^{p/q} = a^{1/q \times p} = \left(a^{\frac{1}{q}}\right)^p$ is positive integer, q ≠ 0

$= a^{1/q} \times a^{1/q} \times \ldots\ldots\ldots\ldots p$ times

- $a^m = a^n \Rightarrow m = n\ when\ a \neq 0, 1$
- $a^m = b^m \Rightarrow a = b$

Ex: $\left(-\frac{1}{343}\right)^{-\frac{2}{3}}$

Sol. $\left(-\frac{1}{343}\right)^{-\frac{2}{3}} = \left(-\frac{1}{7^3}\right)^{-2/3} = (-7^{-3})^{-2/3}$

$= (-7)^{-3 \times -\frac{2}{3}} = (-7)^2 = 49$

Ex: $3^{-3} + (-3)^3$

Sol. $3^{-3} + (-3)^3 = \frac{1}{3^3} + (-3)^3 = \frac{1}{27} - 27$

$= \frac{1-729}{27} = -\frac{728}{27}$

Ex: If $2^{2x-1} = \frac{1}{8^{(x-3)}}$, then x =?

Sol. $2^{2x-1} = \frac{1}{8^{(x-3)}} \Rightarrow 2^{2x-1} = \frac{1}{2^{3(x-3)}}$

$\Rightarrow 2^{2x-1} = \frac{1}{2^{3x-9}}$

$\Rightarrow (2^{2x-1})(2^{3x-9}) = 1$

$\Rightarrow 2^{(2x-1)+(3x-9)} = 1$

$\Rightarrow 2^{5x-10} = 1 \Rightarrow 2^{5(x-2)} = 1$

$\Rightarrow 2^{5(x-2)} = 2^{\circ}$

$\Rightarrow$ x – 2 = 0 $\Rightarrow$ x = 2

Type – II

Surd: If a rational and n is a positive integer and $a^{1/n}$ = $\sqrt[n]{a}$ is Irrational, then $\sqrt[n]{a}$ is called "surd of order n" or "nth root of a" For the surd $\sqrt[n]{a}$, n is called the surd – index or the order of the surd and "a" is called the radicand. The symbol "$\sqrt{\ }$ is called the surd sign or radical.

Ex. $\sqrt{5}$ is a surd of order 2 or square root of 5.

$\sqrt[3]{6}$ is surd of order 3 or cube root of 6.

$\sqrt{6+5}$ is not a surd as

6 + $\sqrt{5}$ is not a rational number.

- Every surd is an irrational number but every irrational number is not a surd.
- In the surd $a\sqrt[n]{b}$, a and b are called factors of the surd.

Ex. $3\sqrt{5}, 2\sqrt{7}, 5\sqrt[3]{7}$

Quadratic surd: A Surd of order 2 (i.e. $\sqrt{a}$) is called a quadratic surd.

Ex: $\sqrt{2} = 2^{1/2}$ is a quadratic surd but $\sqrt{4} = 4^{1/2}$ is not a quadratic surd because $\sqrt{4} = 2$ is a rational number.

Therefore $\sqrt{4}$ is not a surd.

Cubic Surd: A surd of order 3(i.e. $\sqrt[3]{a}$) is called a cubic surd.

Ex. $\sqrt[3]{9}$ is a cubic surd but $\sqrt[3]{27}$ is not a surd because $\sqrt[3]{27}$ = 3 is rational number.

Important Formula Based on Surds:

(i) $\sqrt[n]{a^n} = a$

(ii) $\sqrt[n]{ab} = \sqrt[n]{a}.\sqrt[n]{b}$

(iii) $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ $and \frac{k\sqrt[n]{a}}{l\sqrt[n]{b}} = \frac{K}{l}\sqrt[n]{\frac{a}{b}}$

(iv) $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} = \sqrt[n]{\sqrt[m]{a}}$

(v) $\left(\sqrt[n]{a^m}\right) = (a)^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}$

(vi) $\sqrt{a} \times \sqrt{a} = a$

(vii) $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ and k. $\sqrt[n]{a} \times l.\sqrt[m]{b} = kl.\sqrt[n]{a}.\sqrt[m]{b} = kl.\sqrt[mn]{a^m b^n}$

(viii) $\sqrt{a^2 b} = a\sqrt{b}$

(ix) $\left(\sqrt{a} + \sqrt{b}\right)^2 = a + b + 2\sqrt{ab}$

(x) $\left(\sqrt{a} - \sqrt{b}\right)^2 = a + b - 2\sqrt{ab}$

(xi) $\left(\sqrt{a} + \sqrt{b}\right) \times \left(\sqrt{a} - \sqrt{b}\right) = a - b$, where a and b are positive rational numbers.

Ex. The surd $\sqrt[4]{3 \times 5^4}$ is not in its simplest form since the number under the radical sign had factor 5^4 . its index is equal to the order of the surd. Its simplest form:

$\sqrt[4]{3 \times 5^4} = \sqrt[4]{3}.\sqrt[4]{5^4} = (\sqrt[4]{3})(5) = 5.(\sqrt[4]{3})$

Similar or like Surds: Surds having same irrational factors are called "similar or like surd".

Ex.: $\sqrt[3]{3}$, $7\sqrt{3}, \frac{2}{5}\sqrt{3}, \sqrt{3}$ etc. are similar surds.

Unlike surds: Surds have non - common irrational factors are called "unlike surds".

Ex. $3\sqrt{3}, 5\sqrt{2}, 6\sqrt{7}$ etc. are unlike surds.

Type III

Comparison of Surds: (i) If two surds are of the same order, then the one whose radicand is larger, is the larger of the two.

Ex. $\sqrt[3]{19} > \sqrt[3]{15}, \sqrt{7} > \sqrt{5}, \sqrt[3]{9} > \sqrt[3]{7}$ etc.

(ii) If two surds are distinct order, we change them into the surds of the same order.

This order is L.C. M. of the orders of the given surds.

Ex.: Which is larger $\sqrt{2}$ *of* $\sqrt[3]{3}$?

Sol. Given surds are of order 2 & 3 respectively whose L.C.M is 6.

Convert each into a surd of order 6, as show below:

$\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{1}{2}\times\frac{3}{3}} = 2^{\frac{3}{6}} = (2^3)^{\frac{1}{6}}$

$= (8)^{1/6} = \sqrt[6]{8}$

$= 3^{\frac{1}{3}} = 3^{\frac{1}{3}\times\frac{2}{2}} = 3^{\frac{2}{6}}\ (9)^{\frac{1}{6}}$

$= \sqrt[6]{9}$

Clearly, $\sqrt[6]{9} > \sqrt[6]{8}, so\ \sqrt[3]{3} > \sqrt{2}$

Type – IV

(a) If $y = \sqrt{x + \sqrt{x + \sqrt{x + \cdots \ldots \ldots \ldots .\infty}}}$

then, $y = \frac{1+\sqrt{1+4x}}{2}$

Ex.: $y = \sqrt{7 + \sqrt{7 + \sqrt{7 \ldots \ldots \ldots .\infty}}}$

Sol. $y = \frac{1+\sqrt{1+4x}}{2}$

Here, x = 7

then $y = \frac{1+\sqrt{1+4\times7}}{2}$

$= \frac{1+\sqrt{29}}{2}$

$\therefore \sqrt{29}$ lies between 5 or 6

So, $y = \frac{1+5}{2} = 3$

or, $y = \frac{1+6}{2} = 3.5$

So, 3 < y < 3.5 is correct.

Type – V

Square – root of an irrational number:

As we know that, $(a + b)^2 = (a^2 + b^2) + 2ab$

$\therefore (\sqrt{2} + \sqrt{3})^2 = \underbrace{5}_{(a^2+b^2)} + \underbrace{2\sqrt{6}}_{(2ab)}$

$\therefore 5 + 2\sqrt{6} = 5 + \underbrace{2\sqrt{2}\sqrt{3}}_{(2ab)}$

$\therefore a = \sqrt{2}\ \&\ b = \sqrt{3}$

$\&\ a^2 + b^2 = 5$

$\therefore 5 + 2\sqrt{6} = (\sqrt{2} + \sqrt{3})^2$

$\Rightarrow a + b = \sqrt{5 + 2\sqrt{6}} = \sqrt{2} + \sqrt{3}$